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PG Semester I

Paper – CC-2

Unit – 4

Topic – Example of order of an element of a group

Example 1: Show that if G is an abelian group, then for all $a, b \in G$ and all integers n

$$(ab)^n = a^n b^n$$

Sol^n → (i) Let $n = 0$

then, $(ab)^0 = e$

Also, $a^0 b^0 = e \cdot e = e$

$\therefore (ab)^0 = a^0 b^0$

(ii) let $n > 0$

If $n = 1$, then $(ab)^1 = ab = a^1 b^1$

Let us suppose our result is true for $n = r$

i.e. $(ab)^r = a^r b^r$,

Then $(ab)^{r+1} = (ab)^r \cdot ab$

$= a^r b^r ab = a^r a b^r b$

$= a^{r+1} b^{r+1}$ $(\because ab^r = b^r a)$

Then, by mathematical induction for all $n > 0$

$$(ab)^n = a^n \cdot b^n$$

(iii) Let $n < 0$
Let $n = -r$, where r is a positive integer.

Then, $(ab)^n = (ab)^{-r} = [(ab)^r]^{-1}$

$= [a^r b^r]^{-1}$

$= [b^r a^r]^{-1}$ $\quad (\because a^m b^m = b^m a^m)$

$= [a^r]^{-1} [b^r]^{-1}$ $\quad [\because (ab)^{-1} = b^{-1} a^{-1}]$

$= a^{-r} b^{-r} = a^n b^n$

Example. For any two elements a and b of a group G, show that G is abelian iff $(ab)^2 = a^2 b^2$.

Sol^n:- Let us first suppose that, G is abelian.
So that $ab = ba \; \forall \; a, b \in G$

Consider $(ab)^2 = (ab)(ab) = a(ba)b$
(By associativity)

$= a(ab)b$ (By commutativity)

$= (aa)(bb)$ (By associativity)

$= a^2 b^2$

Thus $(ab)^2 = a^2 b^2 \; \forall \; a, b \in G$.

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~~Thursday~~ **Conversely** Let $(ab)^2 = a^2b^2 \ \forall\ a,b \in G$.

To show $ab = ba$

consider $(ab)^2 = a^2b^2 \Rightarrow (ab)(ab) = (aa)(bb)$

$\Rightarrow a(ba)b = a(ab)b$ (By associativity)

$ba = ab$

(By left and right cancellation law)

Thus, we have

$ab = ba \quad \forall\ a,b \in G$

Hence G is abelian.

Important Examples

(i) If every element of a group has its own inverse, then group must be abelian.

(ii) If number of elements in a group is less than or equal to 4, then group is abelian.

(iii) If $o(a) = o(b) = o(ab) = 2$

Then, $ab = ba$

i.e. Group G is abelian.

(iv) If G is a group such that $(ab)^m = a^m b^m$ for three consecutive integers m $\forall a, b \in G$, then G is abelian.

(v) Group G is abelian iff

$$(ab)^{-1} = a^{-1}b^{-1} \quad \forall a, b \in G$$

(vi) If in a group G,

$$a^5 = e, \quad aba^{-1} = b^2$$

for $a, b \in G$, then

$O(b) = 1$ if $b = e$ and $o(b) = 31$ if $b \neq e$

where, e is an identity element of group.

(vii) A group of even order has atleast one element of order 2.

(viii) Let $G = \{a + b\sqrt{2}, a, b \in \mathbb{Q}\}$, then G is an infinite abelian group w.r.t binary operation '$+$'.

$G = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} ; a, b \in R \text{ such that } a^2 + b^2 \neq 0 \right\}$ form an infinite abelian

group under the operation of matrix
multiplication.